The Numerical Performance of Wavelets and Reproducing Kernels for PDE's

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The use of wavelets for solving PDE's has promised many advantages ...

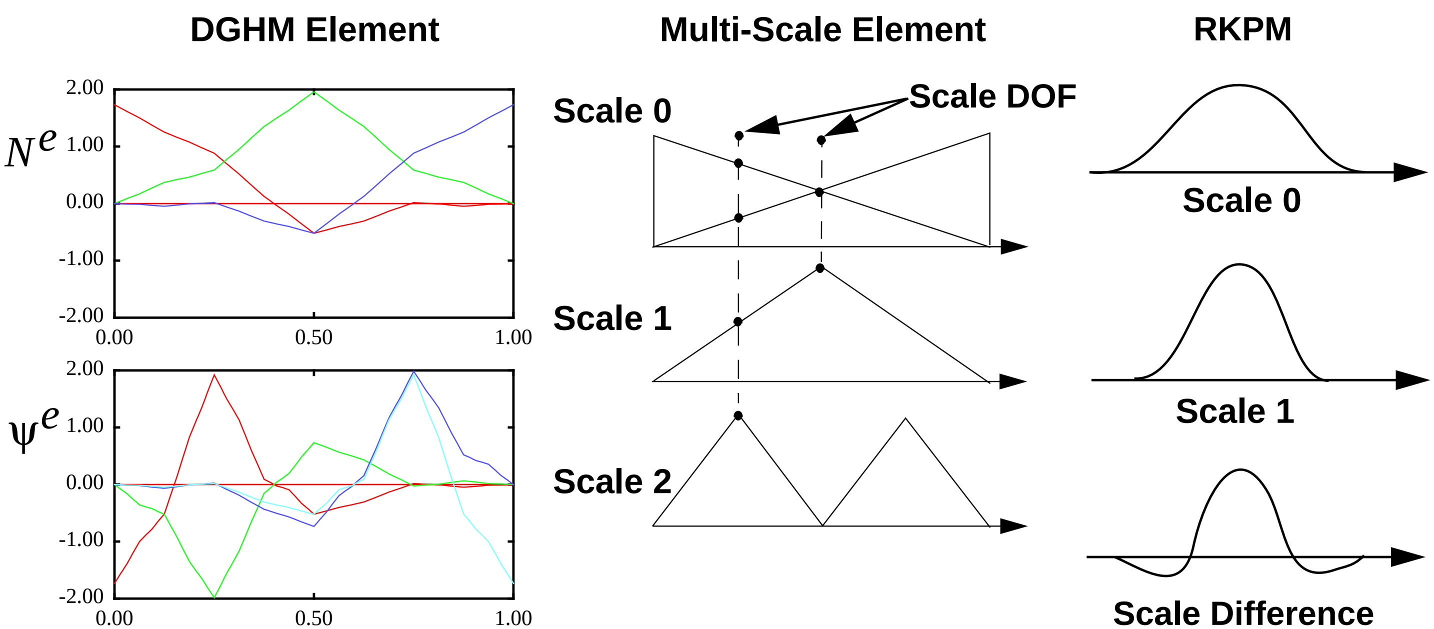
- Convergence rates superior to finite difference, finite element methods
- Built-in and "automatic" h-adaptivity
- "Scale-matched" hierarchical PDE solutions
- Combined advantages of spectral and finite difference methods

"The theoretical and numerical results suggest that for the above class of problems [ODE's and PDE's] wavelets provide a robust and accurate alternative to more traditional methods such as finite differences and finite elements", Glowinski, et al., 1990

"The convergence rates of wavelet solutions are examined and they are found to compare extremely favorably to the finite difference solutions", Amartunga, et al., 1994

This Work is Attempting to Quantify the Numerical Performance of RKPM and Wavelet Bases

- Numerical performance includes: truncation error, consistency, stability, rate of convergence, dispersive character, and spatial adaptivity



Numerical analysis has quantified the performance of the DGHM wavelet element

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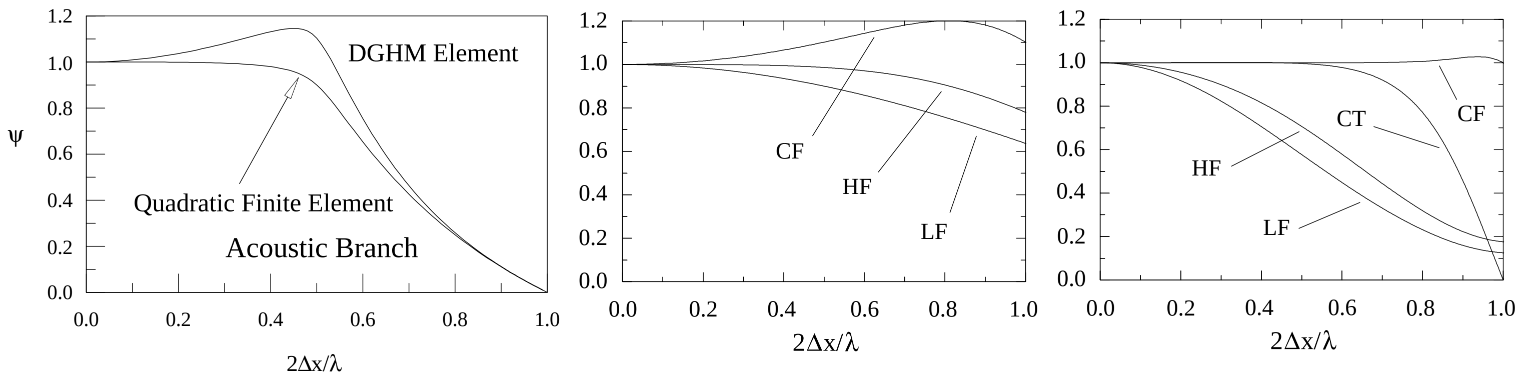
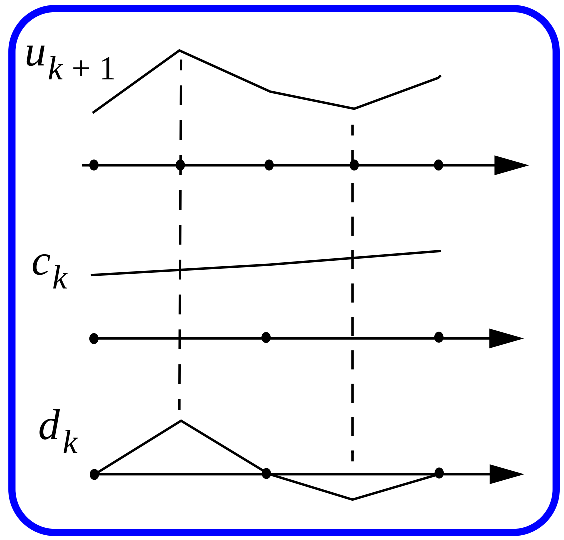
- DGHM delivers $O(h^2)$ accuracy despite its quadratic appearance
- Fractal multi-scaling functions can only represent $\{1, x\}$
- The computational cost is that of a quadratic element (+)
- DGHM exhibits inferior dispersive behavior for the second-order wave equation

DGHM **FEM** **RKPM**



DGHM has provided insight into wavelet-based multi-level solution algorithms

- $-u'' = f, 0 \le x \le 1$ with $u(0) = \alpha, \quad u(1) = \beta$
- $u_{k+1} = \sum \phi_k c_k + \sum \psi_k d_k$
- $\begin{bmatrix} K_k^{\phi\phi} & K_k^{\phi\psi} \\ K_k^{\psi\phi} & K_k^{\psi\psi} \end{bmatrix} \begin{bmatrix} c_k \\ d_k \end{bmatrix} = \begin{bmatrix} f_k^{\phi} \\ f_k^{\psi} \end{bmatrix}$
- Coarse-grid solve: $K_k^{\phi\phi} c_k^0 = f_k^{\phi}$



- Due to orthogonality: $T_k^{\phi\psi} = [K_k^{\phi\phi}]^{-1} K_k^{\phi\psi}$ may be assembled
- Refinement: $[K_k^{\psi\psi} - K_k^{\psi\phi} T_k^{\phi\psi}] d_k = f_k - K_k^{\psi\phi} c_k^0$
- Correction: $c_k = c_k^0 - T_k^{\phi\psi} d_k$

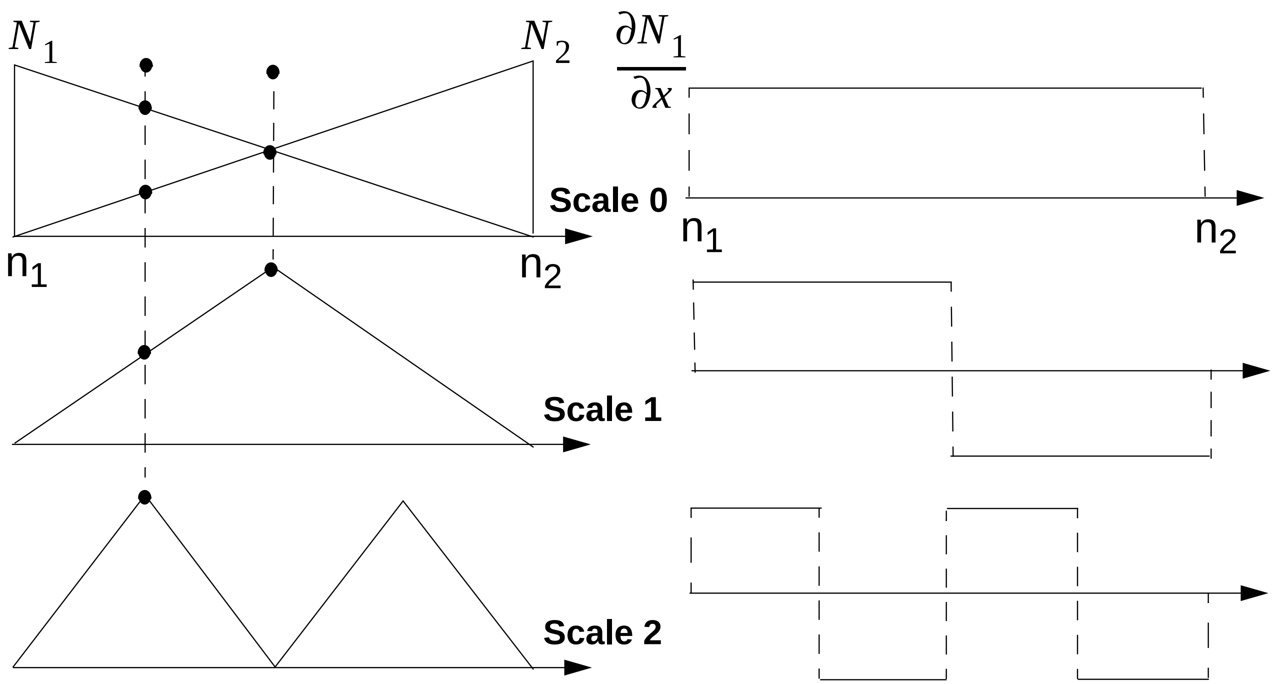
Multi-level solution strategy exhibits uniformly bounded condition number for multi-scale DOF

	DGHM Multi-Wavelet	Quadratic Multi-Wavelet
Element Level k	$cond([K_k^{\Psi\Psi} - K_k^{\Psi\phi} T_k^{\phi\Psi}])$	$cond([K_k^{\Psi\Psi} - K_k^{\Psi\phi} T_k^{\phi\Psi}])$
0	2.3533	5.5676
1	2.5497	5.7578
2	2.6057	5.7945
3	2.6201	5.8063
4	2.6238	5.8093

- Stationary iterative techniques may be used to solve for the multi-scale DOF (wavelet coefficients)
  - Spectral radius for Jacobi iteration: $\rho \approx 0.135$
  - Spectral radius for Gauss-Seidel iteration: $\rho \approx 0.018$

Multi-scale "wavelet" finite element relies upon an H^1 stable basis for elliptic problems

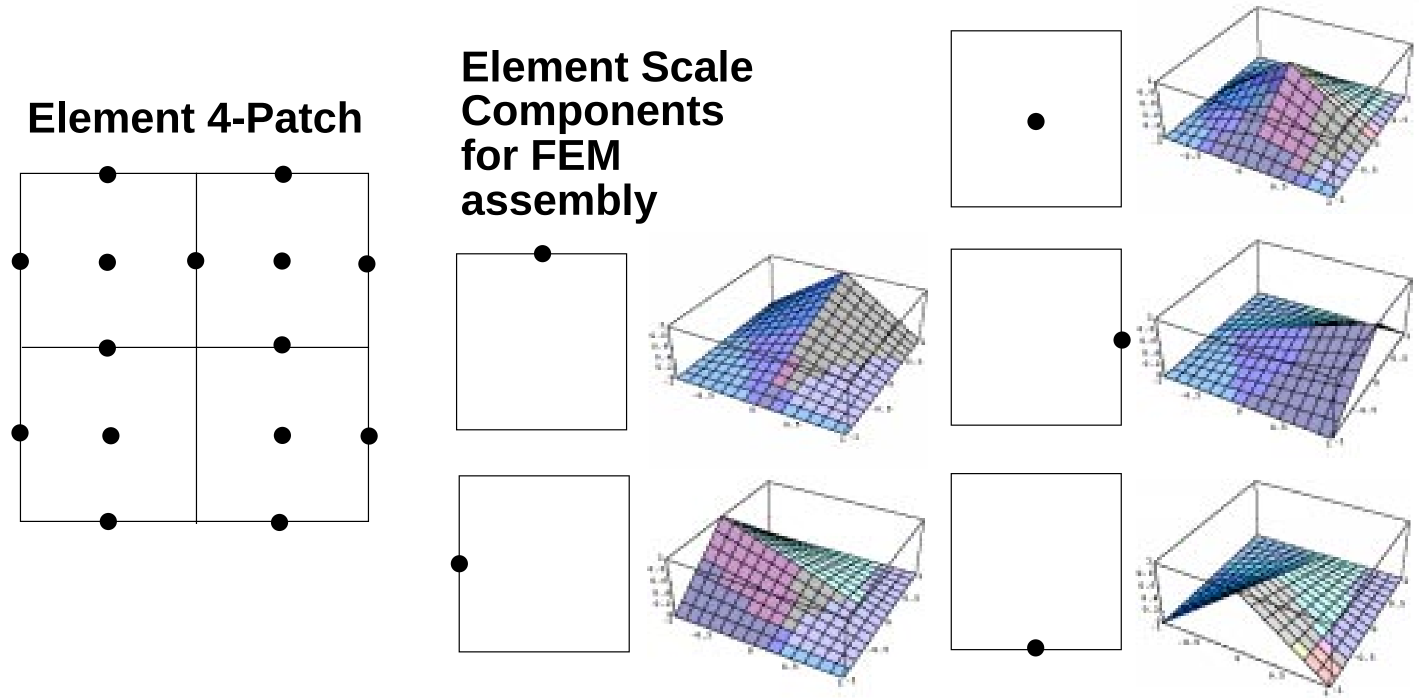
- $-u'' = f,\ 0 \le x \le 1$ with $u(0) = \alpha, \quad u(1) = \beta$
- Condition number is mesh independent: $\kappa = O(1)$



"Stiffness"

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 & 0 \\ 0 & 0 & K^1 & 0 & 0 \\ 0 & 0 & 0 & K^2 & 0 \\ 0 & 0 & 0 & 0 & K^2 \end{bmatrix}$$

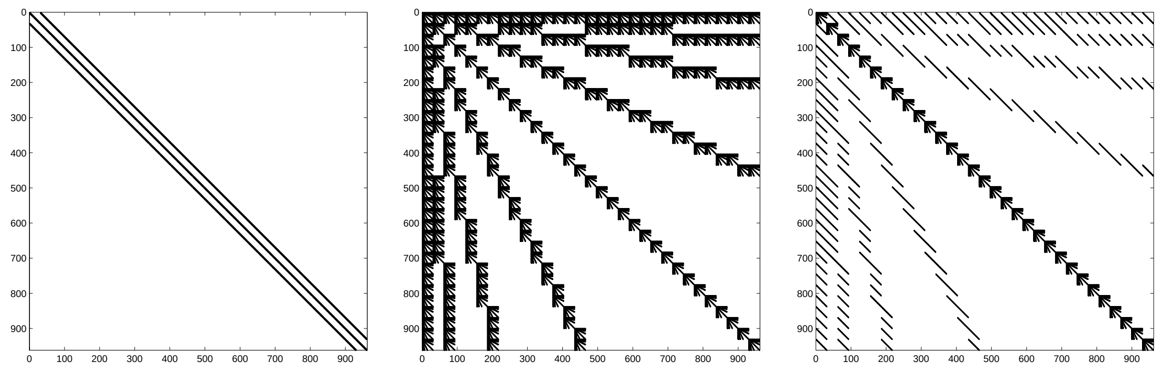
Two-Dimensional Multi-scale element extends the Schauder basis



- $\kappa = O(J^2)$ for 2-D, $O(2^J)$ for 3-D, J = number of levels
- Multi-scale element is compatible with h-adaptive codes such as ALEGRA and MPSALSA

The Schauder Basis Yields Finger-Diagonal Mass and Stiffness Operators

- Reduced condition number is offset by the finger-diagonal matrix structure for an $N \times N$ mesh — Schauder wins for $N \times N > 10^5$
  - Schauder Mass: $N_{NZ} = ((2k-1)(N+1)+3)^2$
  - Schauder Stiffness: $N_{NZ} = N((4k-3)(N+1)+7)$
  - FEM Storage: $N_{NZ} = (3N-2)^2$



FEM **Schauder - Mass and Stiffness**

Row-Column Sum Lumping Reduces Storage and Simplifies Multi-Level Solution Scheme

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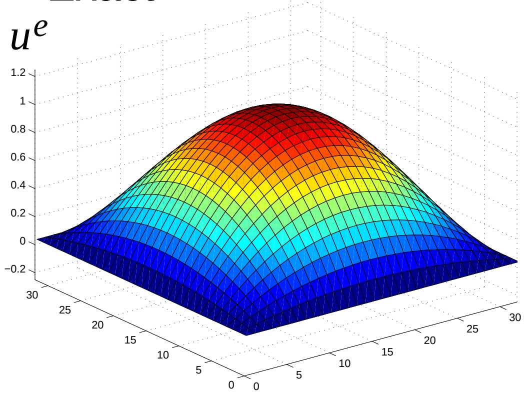
- $-\varepsilon\nabla^2 u + u = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma$

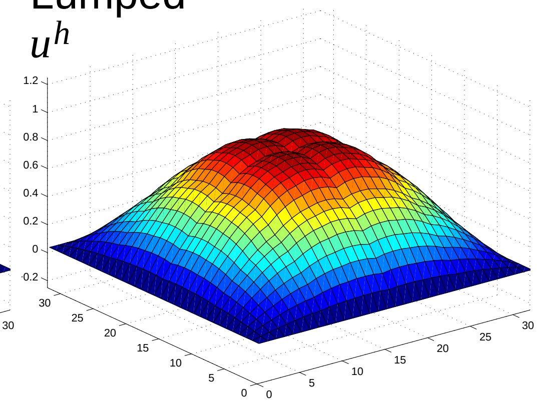
Exact

u^e



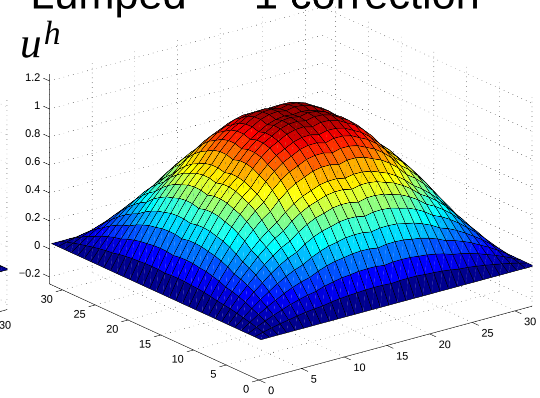
Lumped

u^h



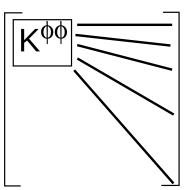
Lumped — 1 correction

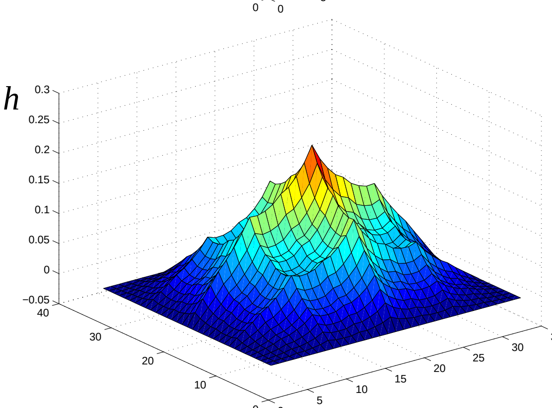
u^h

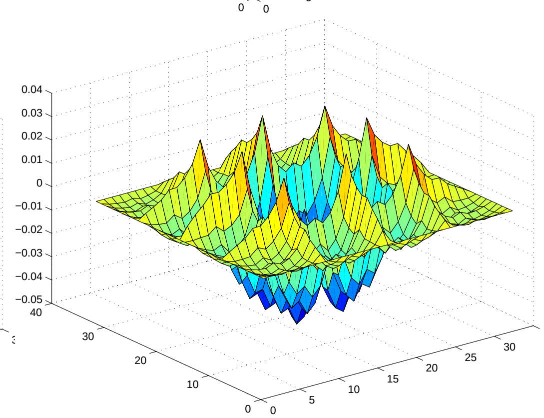


Error: $u^e - u^h$

K=







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Summary and Conclusions

- DGHM (and related) multi-wavelets are not a good choice for a Hierarchical basis for solving PDE's
- The Schauder basis is a good prototype of the IDEAL Hierarchical basis but may require the use of ad-hoc row-column lumping for multi-dimensions
  - The multi-scale elements (Schauder basis) have great potential for developing fast elliptic solver
- The construction of hierarchical bases that yield uniformly bounded condition number, i.e., are stable, for arbitrary PDE's remains an open issue

At this time, the use of wavelet bases for PDE's remains a research topic centered squarely in the Mathematics community

Multiresolution analysis breaks $L^2(R)$ into a sequence of nested subspaces

- The multi-scale representation of a function relies on a series of nested subspaces
  - $\{0\} \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots \subset L^2(R)$
  - The closed union is dens in $L^2(R)$, $\bigcup_{j \in Z} V_j = L^2(R)$
  - The intersection is the trivial set, $\bigcap_{j \in Z} V_j = \{0\}$
  - The spaces in a multiresolution analysis are related by a scaling law, $f(x) \in V_j \Leftrightarrow f(2x) \in V_j$
  - Each subspace is spanned by integer translates of the scaling function